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Structural Analysis and Uncertainty of Orbital Angular Momentum Operators in Spherical Representation: Projection Approach, Spherical Tensor, and Their Application to Atomic Systems

Hanan Zaki Alhusni, Mohamad Wahdiansyah Arrahmat*, Munasir, Habibi
Universitas Negeri Surabaya, Surabaya, Indonesia

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ABSTRACT

This research is a literature review that comprehensively discusses the properties of the orbital angular momentum operator in spherical representation. The main focus lies in the analysis of the mathematical structure of the operator through the projection operator and spherical tensor approach, as well as its physical implications in various quantum systems. This study also evaluates the limitations of the uncertainty principle that arises from the commutator relation between the components of the angular momentum operator, and how this uncertainty depends on the specific quantum state. The results show that the spherical representation provides a natural framework for modeling rotational symmetry in atomic and electromagnetic systems. The projection operator allows the construction of structured wave functions, while the spherical tensor expands the capabilities of operator decompositions and transformations in Hilbert space. Furthermore, orbital angular momentum is shown to play a crucial role in the formation of multipolar electromagnetic fields and the energy structure of atoms. This study strengthens the conceptual understanding of the orbital angular momentum operator and opens up further research directions in precision quantum control and optical systems engineering.

INTRODUCTION

In quantum mechanics, orbital angular momentum is one of the most fundamental physical quantities because it acts as a generator of rotation in Hilbert space and is closely related to the symmetries of quantum systems, such as rotation and conservation of angular momentum (Edmonds, 1996). This quantity plays a crucial role in explaining the structure and dynamics of microscopic systems, particularly atomic and molecular systems, such as the distribution of electron orbitals, spin-orbit coupling, and transition selection rules (Weissbluth, 2012). In this context, the spherical representation of the angular momentum operator using spherical harmonic basis functions provides an efficient mathematical framework because it naturally aligns with the rotational symmetry of physical systems, which is common in centrally charged systems such as the hydrogen atom and diatomic molecules (Devanathan, 2005).

One of the main challenges in understanding the nature of the angular momentum operator is the complexity of its mathematical structure, especially when developed in the spherical wave function basis, as it involves complex relationships between radial and angular components and symmetry transformation rules (Devanathan, 2005). The approach using projection operators and spherical tensor operators has proven to be a powerful method for representing rotational transformations in Hilbert space and simplifying the separation of spatial components (Efimov, 1979). This method not only clarifies the internal structure of the angular momentum operator but is also very useful for the analysis of multiparticle systems and systems with certain symmetries, such as in atomic and molecular physics (Louck, 2006).

On the other hand, fundamental limitations in the measurement of angular momentum components, reflected in the uncertainty relations, raise important questions about the limits of precision in quantum experiments and the information that can be obtained from a physical system (Dammeier, Schwonnek, & Werner, 2015). The study of these uncertainty relations for the angular momentum operator has led to a formal description of the statistical limits for simultaneous observation of the components (Franke-Arnold, Barnett, & Yao, 2004). The relevance of this study is further enhanced in the context of quantum technology, where high-precision state manipulation based on angular momentum, such as in spin- and orbital-based quantum information processing, is becoming a key requirement (Leach et al., 2010).

Furthermore, advances in quantum optics and electromagnetic field manipulation have paved the way for the application of orbital angular momentum concepts to various advanced technological domains, including quantum optics, photonic communications, and the control of trapped atomic systems (Zhou et al., 2016). In this context, an in-depth analysis of the properties of orbital angular momentum operators is crucial, particularly in designing them for applications in photonic systems, where the topological properties and rotational symmetry of light play a central role (Barnett & Babiker, 2017).

Therefore, this study aims to provide a comprehensive structural analysis of orbital angular momentum operators in spherical representations by reviewing mathematical approaches such as projection operators and spherical tensors, while also exploring uncertainty aspects and their applications in modern atomic and photonic systems.

RESEARCH METHOD

This research uses a literature review approach as the main method. The literature review was conducted by searching, reading, and analyzing various scientific sources such as journals, books, and other publications relevant to the research theme, namely the properties of orbital angular momentum operators in spherical representations. The literature search focused on articles containing keywords such as orbital angular momentum, angular momentum operator, commutation relations, and spherical harmonics. The obtained articles were then reviewed in depth and systematically to identify the theoretical approaches used, review the results of previous research, and draw meaningful conclusions. More than simply summarizing the contents of the literature, this study also aims to uncover conceptual patterns, establish relationships between ideas, and formulate new insights based on the synthesis of the various references reviewed.

RESULTS AND DISCUSSION

This study presents the results of an in-depth literature study of the concept, mathematical structure, and measurement limitations of the orbital angular momentum operator in spherical representations. This discussion covers three main topics: the construction and transformation of operators in spherical space, commutator and uncertainty relations, and applications of orbital angular momentum operators in atomic systems and electromagnetic fields.

Construction of angular momentum operators in spherical representations

Spherical representations provide a highly efficient way to represent angular momentum operators due to their inherent symmetry with respect to rotation. One significant early approach to constructing angular momentum wave functions was through projection operators, as introduced by Löwdin (1964). In this approach, projection operators are used to construct a basis for wave functions that are orthonormal and satisfy the symmetry relations of the SU (2) rotation group, which is the basis for the representation of angular momentum in quantum mechanics.

Furthermore, Lindgren and Morrison (1982) developed a comprehensive theory of spherical tensor operators, which is a crucial component in formalizing angular momentum operators in atomic systems. This spherical tensor operator allows for operator decomposition based on the spherical harmonic degrees $Ylm\ (\theta,\ \phi)$, which then facilitates the analysis of transformations regarding rotations and interactions between particles in the many-electron atomic framework.

Commutator relations and the uncertainty principle

One important property of the angular momentum operator is the commutator relation between its components. The relation $[Li,\ Lj]=i\hbar\epsilon ijkLk$ forms the basis for the emergence of the uncertainty principle in the context of angular momentum, where it is impossible to precisely measure two different components simultaneously. Dammeier, Schwonnek, and Werner (2015) proved new, stricter forms of the uncertainty principle for the angular momentum operator and showed that the uncertainty bound is not universal, as for position and linear momentum, but rather depends strongly on the specific quantum state of the system.

These findings broaden our understanding of the intrinsic limitations in measuring orbital angular momentum, which impact experimental limitations in controlling quantum systems, particularly in systems involving superposition states of angular momentum.

The role of angular momentum in electromagnetic fields and atomic systems

Orbital angular momentum is not limited to atomic contexts but also plays a crucial role in electromagnetic fields, particularly multipolar ones. Tischler, Zambrana-Puyalto, and Molina-Terriza (2012) explain that electromagnetic fields with a multipolar structure can be constructed using spherical harmonic basis functions, which are directly related to the angular momentum operator. Thus, this operator serves as a crucial tool in formulating and classifying light modes that carry orbital angular momentum, such as in optical vortex light systems.

In the context of atomic systems, as summarized in the Springer Handbook of Atomic, Molecular, and Optical Physics (Drake, 2023), the orbital angular momentum operator plays a key role in determining the energy structure of atoms and in their interactions with external fields. Orbital angular momentum determines energy level splitting due to spin-orbit effects and plays a role in the selection rules for photon emission and absorption.

Synthesis and theoretical implications

From this literature review, it can be concluded that the projection operator approach (Löwdin, 1964) and the spherical tensor (Lindgren & Morrison, 1982) provide a powerful formal framework for representing and analyzing orbital angular momentum operators. The commutator relation and the extended uncertainty principle (Dammeier et al., 2015) provide important theoretical limits on experimental measurements, while the relationship of orbital angular momentum to electromagnetic fields (Tischler et al., 2012) and atomic structure (Drake, 2023) broadens its application to various branches of modern physics.

The implications of this study include a deeper understanding of the symmetry of quantum systems, the limits of high-precision quantum measurements, and the development of angular momentum-based technologies such as optical communication, atomic quantum control, and spherical field-based structured systems.

This study has thoroughly examined the properties of orbital angular momentum operators in spherical representations using literature from five primary sources. The following key points can be concluded from this study:

1. The spherical representation provides a natural structure for analyzing angular momentum operators because it aligns with the rotational symmetry underlying many physical systems. This approach allows for more efficient modeling and calculations in quantum state spaces.
2. The projection operator approach developed by Löwdin provides a foundation for systematically constructing angular momentum-structured wave functions, clarifying the role of the SU (2) rotation group in quantum theory.
3. The spherical tensor operator, as formulated by Lindgren and Morrison, provides a powerful mathematical tool for operator decomposition and transformation with respect to rotations and is particularly relevant in many-electron atomic systems.
4. The commutator relation between the components of the angular momentum operator yields uncertainty bounds, which, extended through a modern formulation by Dammeier et al., demonstrate that the uncertainty in these systems is state-dependent and not universal.
5. The applications of the angular momentum operator extend to a wide range of physical systems, from atomic structures to multipolar electromagnetic fields. A study by Tischler et al. shows that twisted light modes carry angular momentum characteristics, while references from the Springer Handbook demonstrate direct implications in atomic energy structures and optical transition processes.

Overall, this study confirms that a thorough understanding of the structural properties and uncertainty principle of the orbital angular momentum operator is crucial for bridging the gap between quantum physics theory and application, particularly in the context of atomic systems and quantum optics. Further research can be directed toward developing numerical models or simulations based on this operator in more complex and realistic systems.

CONCLUSION

This study highlights the crucial role of spherical representations in analyzing orbital angular momentum operators, which align naturally with the rotational symmetry of quantum systems. The projection operator approach (Löwdin) and spherical tensor operator (Lindgren & Morrison) provide effective mathematical tools for constructing and transforming wave functions, especially in multi-electron atoms. The commutator relations yield a state-dependent uncertainty principle (Dammeier et al.), emphasizing limitations in precise measurement. Beyond atomic structures, orbital angular momentum is essential in multipolar electromagnetic fields and twisted light technologies. Overall, this research bridges theoretical quantum mechanics with practical applications in quantum optics and atomic control.

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